Euler's Angles

Euler suggested that orientation of rigid body relative to a fixed co-ordate system can be tranformed into the orientation relative to body co-ordindte system by means of three successive rotations by angles φ, θ and ψ in a specific sequence (order). φ, θ, ψ are called Euler's angles.

Let $Oxyz$ be fixed co-ordinate system and the body rotates about fixed point O. Let $OX_1Y_1Z_1$ be body co-ordinate system. (usually principal axes). Suppose initially (at $t=0$) the two system coincident. (aligned with fixed axes)

First Rotation # First rotation is made through angle φ in XY plane of fixed co-ordinate system about axis OZ and it transforms body axes to $OX'Y'Z'$

The matrix of this transformation is

$$R_\varphi = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse of this matrix is

$$R_\varphi^{-1} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The angle φ is called precession angle and it varies from 0 to 2π.

2nd Rotation 2nd rotation is made in counterclockwise in $Y'Z'$-plane about OX' axis through an angle θ, this rotation transforms the axes $X' = X''$, $Y' \to Y''$, $Z' \to Z''$

The matrix of this transformation is given by

$$R_\theta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

The angle θ is called nutation angle and it varies from 0 to π.

3rd Rotation 3rd rotation is mad counter clockwise in $OX''Y''$-plane through ψ and it transforms the axes $X'' = X' \to X_1$, $Y'' \to Y_1$ $Z'' = Z' \to Z_1$

The rotation matrix of this transformation is is

$$R_\psi = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The angle ψ is called body angle and it varies from 0 to 2π

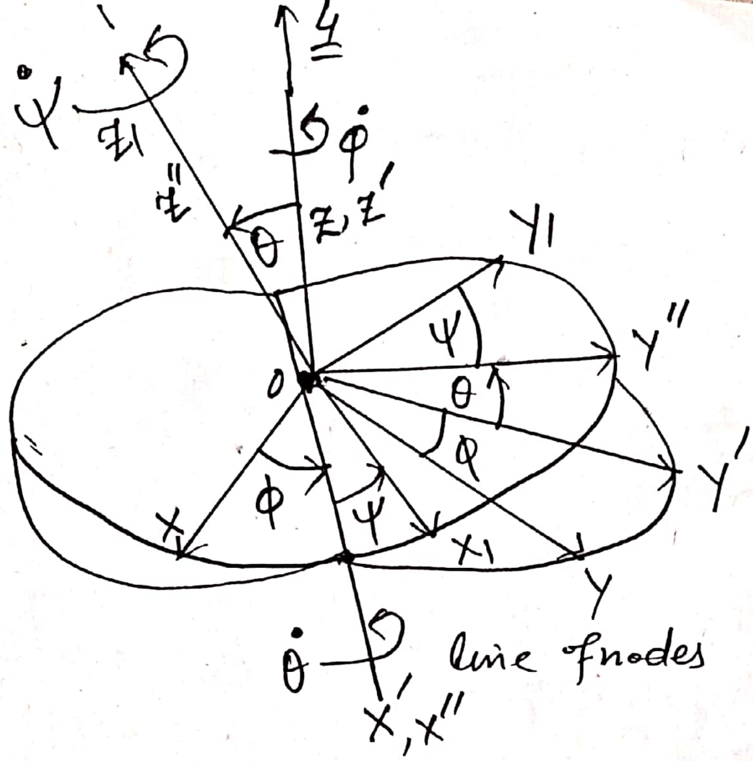
The transformation from the fixed co-ordinate system to the body co-ordinate system $OX_1Y_1Z_1$ is given by rotation Matrix

$$R = R_\psi R_\theta R_\phi$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

As the body rotates the Euler's angles vary with time and their time derivatives $\dot{\theta}$ $\dot{\phi}$ $\dot{\psi}$ give angular speeds about certain axes.



Angular Velocity in terms of Euler's Angles

The angular velocities $\vec{\omega}_\phi = \vec{\dot{\phi}}$, $\vec{\omega}_\theta = \vec{\dot{\theta}}$, $\vec{\omega}_\psi = \vec{\dot{\psi}}$ are directed along following axes

$\vec{\omega}_\phi = \vec{\dot{\phi}}$ along Z'- fixed axis

$\vec{\omega}_\theta = \vec{\dot{\theta}}$ along line of nodes X'' axis

$\vec{\omega}_\psi = \vec{\dot{\psi}}$ along Z_1 (body) axis

vector $\vec{\omega}$ can be expressed as

$$\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi \longrightarrow (1)$$

and magnitudes of $\vec{\omega}_\phi$, $\vec{\omega}_\theta$, $\vec{\omega}_\psi$ are.

$$\left.\begin{aligned} \omega_\phi &= \dot{\phi} \\ \omega_\theta &= \dot{\theta} \\ \omega_\psi &= \dot{\psi} \end{aligned}\right\} \longrightarrow (2)$$

Components of $\vec{\omega}$ in Body Co-ordinate

To obtain components $\omega_1, \omega_2, \omega_3$ of $\vec{\omega}$ in body co-ordinate system we take Contributions to angular velocity in body systemen of $\vec{\omega}_\phi$, $\vec{\omega}_\theta$, $\vec{\omega}_\psi$ i.e.

$$\vec{\omega}(b) = \vec{\omega}_\phi(b) + \vec{\omega}_\theta(b) + \vec{\omega}_\psi(b)$$

For $\vec{\omega}_\phi(b)$

$\vec{\omega}_\phi(b)$ denotes contribution to angular velocity in body axes due to rotation Through angle ϕ

$\because \vec{\omega}_\phi = \vec{\dot{\phi}} = (0, 0, \dot{\phi})$ in spaces ~~and~~ axes. So

$$\vec{\omega}_\phi(b) = R_\psi R_\theta R_\phi \vec{\omega}_\phi$$

$$= R_\psi R_\theta \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$= R_\psi R_\theta \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$= R_\psi \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\phi}\sin\theta \\ \dot{\phi}\cos\theta \end{bmatrix}$$

$$\vec{\omega}_\phi(b) = \begin{bmatrix} \dot\phi \sin\theta \sin\psi \\ \dot\phi \sin\theta \cos\psi \\ \dot\phi \cos\theta \end{bmatrix}$$

$\vec{\omega}_\theta(b)$

$\vec{\omega}_\theta$ is along OX' and therefore in $OX'Y'Z'$

$$\vec{\omega}_\theta = \begin{bmatrix} \dot\theta \\ 0 \\ 0 \end{bmatrix}$$

Its transform in body co-ordinates is

$$\vec{\omega}_\theta(b) = R_\psi R_\theta \vec{\omega}_\theta = \begin{bmatrix} \dot\theta \cos\psi \\ -\dot\theta \sin\psi \\ 0 \end{bmatrix}$$

$$\vec{\omega}_\psi(b) = \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \dot\phi \sin\theta \sin\psi \\ \dot\phi \sin\theta \cos\psi \\ \dot\phi \cos\theta \end{bmatrix} + \begin{bmatrix} \dot\theta \cos\psi \\ -\dot\theta \sin\psi \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\psi \end{bmatrix}$$

$$\omega_1 = \dot\phi \sin\theta \sin\psi + \dot\theta \cos\psi$$
$$\omega_2 = \dot\phi \sin\theta \cos\psi - \dot\theta \sin\psi$$
$$\omega_3 = \dot\phi \cos\theta + \dot\psi$$

$\Rightarrow$ In Body System.

Euler's. Equations of Motion in Force Field

The rotational part of the motion of any body under the action of torque with reference to inertial frame (S frame) or fixed system is given by

$$\underline{N} = \left(\frac{d\underline{L}}{dt}\right)_{fixed} \longrightarrow ①$$

Where $\underline{N}$ is the net applied torque and $\underline{L}$ is angular momentum given by

$$\underline{L} = I\underline{\omega}$$

When the moment of inertia is calculated about an axis fixed in space it would go on changing as the body rotates. In order to avoid this complication and we employ a co-ordinate system fixed in body and rotates with it. The angular velocity of the body and the co-ordinate system is one and the same.

(there is an exception, if two of the three principal axes I_1, I_2, I_3 are equal, then co-ordinates axes need not ~~to~~ be fixed in the body to be principal axes). Again angular momentum is most conveniently expressed in terms of principal axes. Therefore we take co-ordinates axes body principal axes.

Suppose the body rotates about point O, fixed in space and OXYZ be body co-ordinates axes which are principal axes.

Let $\underline{L} = L_1\hat{i} + L_2\hat{j} + L_3\hat{k}$

$\underline{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$

where the components of $\underline{\omega}$ are taken along the directions of principal axes

Now equation ① in body system (or rotating) is

$$\underline{N} = \left(\frac{d\underline{L}}{dt}\right)_{fixed} = \left(\frac{d\underline{L}}{dt}\right)_{body} + \underline{\omega} \times \underline{L} \rightarrow ②$$

Since the external torque does not depend on its reference frame, every quantity in ② is in the body reference frame. (We may drop the suffix body)

As $[\underline{L}] = [I][\underline{\omega}]$ and axes are principal axes. So

$$\underline{L} = I_1 \omega_1 \hat{i} + I_2 \omega_2 \hat{j} + I_3 \omega_3 \hat{k}$$

where I_1 I_2 I_3 remain constant:

$$\left(\frac{d\underline{L}}{dt}\right)_{body} = I_1 \dot{\omega}_1 \hat{i} + I_2 \dot{\omega}_2 \hat{j} + I_3 \dot{\omega}_3 \hat{k}$$

② becomes

$$N_1 \hat{i} + N_2 \hat{j} + N_3 \hat{k} = I_1 \dot{\omega}_1 \hat{i} + I_2 \dot{\omega}_2 \hat{j} + I_3 \dot{\omega}_3 \hat{k} + \underline{\omega} \times \underline{L}$$

$$= I_1 \dot{\omega}_1 \hat{i} + I_2 \dot{\omega}_2 \hat{j} + I_3 \dot{\omega}_3 \hat{k} + (\omega_2 L_3 - \omega_3 L_2)\hat{i} + (\omega_3 L_1 - \omega_1 L_3)\hat{j} + (\omega_1 L_2 - \omega_2 L_1)\hat{k}$$

Comparing

$$I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2 \omega_3 = N_1$$

$$I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3 \omega_1 = N_2$$

$$I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1 \omega_2 = N_3$$

These equations are called Euler's equations of motion of a rigid body along the principal axes of the body.

These are also called Euler's Dynamical Equations

Note The motion of a rigid body depends upon the structure of the body only through the three numbers I_1, I_2, I_3 ie principal moments of inertia. Thus any two bodies with same principal moments of inertial move in exactly the same manner regardless of the fact that they may have quite different shapes. The simplest geometrical shape that already having three given principal moments of inertia may possess is homogenous ellipsoid. The motion of any rigid body can therefore represented by the motion of equivalent ellisoid. Such representation of motion was originated by Poinsot. The poinsot construction is sometimes useful for depicting the motion of a rigid by geometrically.

Euler's Equations For Free Motion of Rigid Body

OR

Euler's Equations for Torque Free Motion

When no external force is applied, the motion of body is called force free motion. In this case no external Torque is acting on the body and

$\therefore \underline{N} = \underline{0} \Rightarrow N_1 = N_2 = N_3 = 0$

and Euler's equations of Motion becomes

$$I_1 \dot{\omega}_1 - (I_2 - I_3)\,\omega_2 \omega_3 = 0$$

$$I_2 \dot{\omega}_2 - (I_3 - I_1)\,\omega_1 \omega_3 = 0$$

$$I_3 \dot{\omega}_3 - (I_1 - I_2)\,\omega_1 \omega_2 = 0$$

Note we note that according Euler's Equations, even in the absence of a torque the angular velocity can change because

$N = 0$

$\Rightarrow \frac{dL}{dt} = 0 \Rightarrow L = $ Costat

i.e angular velocities must change in order that angular momentum to remain Constant. That is so unless the rotation takes place about a principal axis. e.g if initial rotation occurs about a principal axis say 1-axis, then we must have $\omega_1 \neq 0$ and $\omega_2 = \omega_3 = 0$ Now Euler's equations are

$$I_1 \dot{\omega}_1 = I_2 \dot{\omega}_2 = I_3 \dot{\omega}_3 = 0$$

or the angular velocity components along principal axis remain Constant, equal to their initial values

Euler's Equations for Body rotating about Fixed Axis

Suppose a body rotates about a fixed axis with Constant angular (momentum) velocity ω. Then

$$\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$$

and Euler's Equations reduces to

$$N_1 = \omega_2 \omega_3 (I_3 - I_2)$$

$$N_2 = \omega_3 \omega_1 (I_1 - I_3)$$

$$N_3 = \omega_1 \omega_2 (I_2 - I_1)$$

These give the components of torque that must be exerted on the body by the Constraints Support.

Question A body rotates about a principal axis with Constant angular velocity ω. Prove that will be no torque on the body at all.

Sol Suppose a body rotates about a principal axis say 1. axis with Constant angular velocity ω

Then

$\omega_1 = \omega \quad \omega_2 = \omega_3 = 0 \Rightarrow \dot{\omega}_1 = 0, \dot{\omega}_2 = \dot{\omega}_3 = 0$

Eulers equations becomes

$N_1 = 0 = N_2 = N_3$

$\Rightarrow$ There is no torque at all.

K.E and Eulers Equations

Q: using Eulers Equations of motion find the rate of change of rotational K.E.

Sol Eulers Equations are.

$$I_1 \dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = N_1 \rightarrow (1)$$

$$I_2 \dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = N_2 \rightarrow (2)$$

$$I_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = N_3 \rightarrow (3)$$

Multiplying (1), (2), (3) with $\omega_1, \omega_2, \omega_3$ respectively

$$I_1 \omega_1 \dot{\omega}_1 - (I_2 - I_3)\omega_1\omega_2\omega_3 = N_1\omega_1$$

$$I_2 \omega_2 \dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_2\omega_3 = N_2\omega_2$$

$$I_3 \omega_3 \dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2\omega_3 = N_3\omega_3$$

Adding these equation

$$I_1 \omega_1 \dot{\omega}_1 + I_2 \omega_2 \dot{\omega}_2 + I_3 \omega_3 \dot{\omega}_3 = N_1\omega_1 + N_2\omega_2 + N_3\omega_3$$

$$I_1 \frac{d}{dt}\left(\frac{1}{2}\omega_1^2\right) + I_2 \frac{d}{dt}\left(\frac{1}{2}\omega_2^2\right) + I_3 \frac{d}{dt}\left(\frac{1}{2}\omega_3^2\right) = \underline{N}\cdot\underline{\omega}$$

$$\frac{d}{dt}\left(\frac{1}{2}I_1\omega_1^2\right) + \frac{d}{dt}\left(\frac{1}{2}I_2\omega_2^2\right) + \frac{d}{dt}\left(\frac{1}{2}I_3\omega_3^2\right) = \underline{N}\cdot\underline{\omega}$$

$$\frac{d}{dt}\left[\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2\right) = \underline{N}\cdot\underline{\omega}$$

$$\frac{d T_{rot}}{dt} = \underline{N}\cdot\underline{\omega}$$

Question Using Eulers Equations of Motion prove that the K.E is constant for torque free motion of body.

Sol For torque free motion of body equations are

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0 \rightarrow (1)$$
$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = 0 \rightarrow (2)$$
$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0 \rightarrow (3)$$

Xing (1), (2) & (3) with $\omega_1, \omega_2, \omega_3$ and adding

$$I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 = 0$$

$$I_1\frac{d}{dt}\left(\frac{1}{2}\omega_1^2\right) + I_2\frac{d}{dt}\left(\frac{1}{2}\omega_2^2\right) + I_3\frac{d}{dt}\left(\frac{1}{2}\omega_3^2\right) = 0$$

$$\frac{d}{dt}\left[\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2\right] = 0$$

$$\frac{dT_{rot}}{dt} = 0$$

$$\Rightarrow T_{rot} = \text{Constant}$$

Theorem In the absence of external torque a body cannot spin with constant angular velocity except about a principal axes.

OR

A body having one point fixed and no external torque has principal axes of inertia equal. Prove that it must rotate with of constant angular velocity.

Proof Eulers dynamical Equations

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0$$
$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = 0$$
$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0$$

Z

Let angular velocity ω be constant. Then

$\dot{\omega}_1 = \dot{\omega}_2 = \dot{\omega}_3 = 0$

$\Rightarrow (I_2 - I_1)\omega_2\omega_3 = 0 \Rightarrow \omega_2\omega_3 = 0$

$(I_3 - I_1)\omega_3\omega_1 = 0 \Rightarrow \omega_3\omega_1 = 0$

$(I_1 - I_2)\omega_1\omega_2 = 0 \Rightarrow \omega_1\omega_2 = 0$

$\Rightarrow \omega_2 = 0$ or $\omega_3 = 0$ and $\omega_3 = 0$ or $\omega_1 = 0$ & $\omega_1 = 0$ or $\omega_2 = 0$

$\Rightarrow$ These are simultaneously satisfied if two components out of these are zero. Hence the rotation axis is principal axis

OR

as $I_1 = I_2 = I_3$

E.E are

$I_1\dot{\omega}_1 = 0$ $\quad$ $I_2\dot{\omega}_2 = 0$ $\quad$ $I_3\dot{\omega}_3 = 0$

$\Rightarrow \dot{\omega}_1 = 0$ $\quad$ $\dot{\omega}_2 = 0$ $\quad$ $\dot{\omega}_3 = 0$

$\omega_1 = $ const $\quad$ $\omega_2 = $ const $\quad$ $\omega_3 = $ const

$\Rightarrow \omega$ is constant.

Question # prove by using Eulers Equations that under force free motion of rigid body the angular momentum remains constant in magnitude

Sol Under force free motion the equation of Eulers are

$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0 \rightarrow (1)$

$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = 0 \rightarrow (2)$

$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = 0 \rightarrow (3)$

Multiplying (1), (2) and (3) with $I_1\omega_1, I_2\omega_2, I_3\omega_3$ respectively and adding, we get

$I_1^2\omega_1\dot{\omega}_1 + I_2^2\omega_2\dot{\omega}_2 + I_3^2\omega_3\dot{\omega}_3 = 0$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2\right) = 0$$

$$\Rightarrow \frac{d}{dt}\left(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2\right) = 0$$

$$\Rightarrow I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2 = \text{Const}$$

Now Angular momentum w.r.t. principal axes is given by

$$\underline{L} = I_1\omega_1\hat{i} + I_2\omega_2\hat{j} + I_3\omega_3\hat{k}$$

$$|\underline{L}|^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$

Thus magnitude of angular momentum remains Constant

Classification of Rigid Bodies

Depending on principal moments of inertia I_1, I_2 I_3 rigid bodies are classified in three types

(a) When $I_1 \neq I_2 \neq I_3$ the body is said to be asymmetric top

(b) when $I_1 \neq I_2 = I_3$, then body is said to be symmetric top

(c) When $I_1 = I_2 = I_3$ the body is said to be spherical top.

Special Cases

(1) when rigid body is restricted in a plane (i.e. plane lamania) third co-ordinate can be taken zero & we get

$$I_3 = I_1 + I_2$$

(2) when particles of body are collinear two of their co-ordinates can be taken zero & we have

$$I_1 = I_2 \qquad I_3 = 0$$

Such a system is known as the rotator and it has only two degrees of freedom. It is so, as the rotation about the axis connecting all points of the rigid body is meaningless.

Force Free Motion of Spherical top

In a spherical top all the three moments of inertia are equal i.e

$$I_1 = I_2 = I_3 = I.$$

The angular momentum of body is

$$\underline{L} = I\omega_1 \hat{i} + I\omega_2 \hat{j} + I\omega_3 \hat{k}$$

The Euler Equation of Motions becomes

$$\dot{\omega}_1 = 0 \qquad \dot{\omega}_2 = 0 \qquad \dot{\omega}_3 = 0$$

$$\Rightarrow \omega_1 = \text{Constant} \qquad \omega_2 = \text{Constant} \qquad \omega_3 = \text{Constant}$$

Hence a spherical top rotates with constant angular speed $\underline{\omega}$ when no external torque is applied.

Force Free Motion of Rigid Rotator

In a rigid rotator all the particles lie along a line and Therefore the two moments of inertia are equal and the third one about the axis of the rotator is zero ($I_1 = I_2 = I$ & $I_3 = 0$). Let the z-axis be taken as axis of rotation. This is also axis of symmetry. The angular momentum of the body is

$$\underline{L} = I(\omega_1 \hat{i} + \omega_2 \hat{j})$$

The Euler's Equations are

$$\dot{\omega}_1 - \omega_2\omega_3 = 0 \qquad \dot{\omega}_2 + \omega_1\omega_3 = 0$$

When $\omega_3 = 0$, we have $\qquad \dot{\omega}_1 = 0, \quad \dot{\omega}_2 = 0$

$$\Rightarrow \omega_1 = \text{Constant} \qquad \omega_2 = \text{Constant} \qquad \underline{L} = I\underline{\omega}$$

Hence, the rotator rotates with a constant angular velocity $\underline{\omega}$ in a direction $\perp$ar to z-axis when no external torque is applies.

Torque Free Motion of Symmetric Top

Q A rigid body which is symmetric about an axis has one point fixed on the axis. Discuss the rotational motion of the body, assuming there are no forces acting other than the reaction force at fixed point

Sol We consider the motion symmetrical top under no external Torque (i.e net torque about any point of body is zero) The c.m of such a body is either at rest or moving with uniform velocity w.r.t fixed axes For any point on the axis of symmetry of such top, the axis itself and two other axes ⊥ar to each other and to it will be the principal axes there. We take origin at the c.m so that body co-ordinate axes are coincident with principal axes there. we choose Z axis along the axis of symmetry so that

$$I_1 = I_2 \neq I_3$$

Euler dynamical Equations are.

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0 \quad \rightarrow (1)$$

$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3 = 0$$

$$I_3\dot{\omega}_3 - (I_2 - I_1)\omega_1\omega_2 = 0$$

putting $I_2 = I_1$

$$I_1\dot{\omega}_1 - (I_1 - I_3)\omega_2\omega_3 = 0 \quad \rightarrow (1)$$

$$I_1\dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3 = 0 \quad \rightarrow (2)$$

$$I_3\dot{\omega}_3 = 0 \quad \rightarrow (3)$$

we consider the case in which velocity vector $\vec{\omega}$ does not lie along a principal axis otherwise, the motion is trivial

from 3 $\dot{\omega}_3 = 0 \quad \because I_3 \neq 0$

$\omega_3(t) = \text{Constant} \rightarrow (4)$

from (1) & (2), we have.

$$\dot{\omega}_1 + \left(\frac{I_3 - I_1}{I_1}\omega_3\right)\omega_2 = 0 \rightarrow (5)$$

$$\dot{\omega}_2 - \left(\frac{I_3 - I_1}{I_1}\right)\omega_3 \omega_1 = 0 \rightarrow (6)$$

Since terms in parentheses are same & constant.

let $\Omega = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3 = \text{a Constant}$

Then we get

$$\dot{\omega}_1 + \Omega \omega_2 = 0 \rightarrow (7)$$

$$\dot{\omega}_2 - \Omega \omega_1 = 0 \rightarrow (8)$$

Differentiating (7) w.r.t t

$$\ddot{\omega}_1 + \Omega \dot{\omega}_2 = 0$$

using (8), we get

$$\ddot{\omega}_1 + \Omega^2 \omega_1 = 0 \rightarrow (9)$$

Diff (2) w.r.t t

$$\ddot{\omega}_2 - \Omega \dot{\omega}_1 = 0$$

using (7), we get

$$\ddot{\omega}_2 + \Omega^2 \omega_2 = 0 \rightarrow (10)$$

Equations (9) & (10) are equations of S.H.M.

Sol of (9) is

$$\omega_1 = \omega_0 \cos \Omega t \rightarrow (11)$$

Diff wrt t

$$\dot{\omega}_1 = -\omega_0 \Omega \sin \Omega t$$

using in (7)

$$-\omega_0 \Omega \sin \Omega t + \Omega \omega_2 = 0$$

$$\omega_2 = \omega_0 \sin \Omega t \rightarrow (12)$$

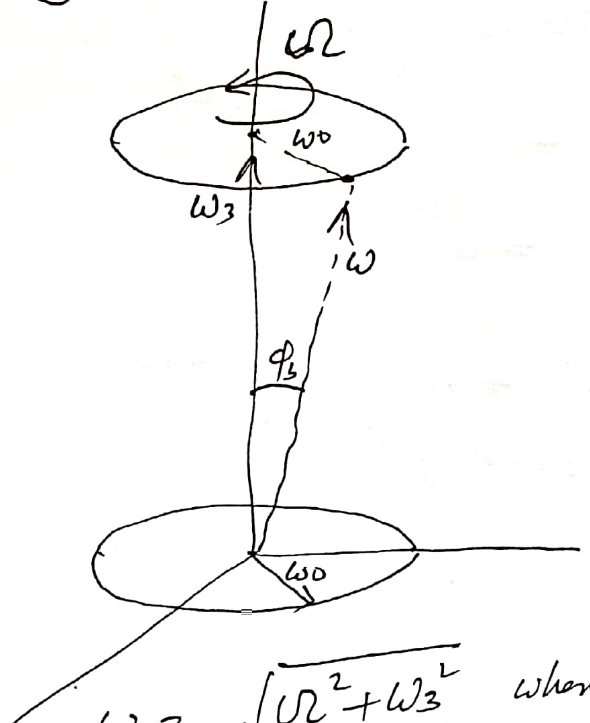
$$= \omega_0 \cos\left(\frac{\pi}{2} - \Omega t\right)$$

Thus ω_1, ω_2 vary harmonically in time with angular frequency Ω and their phases differ by $\pi/2$

(12)

$$|\underline{\omega}|^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$$
$$= \Omega^2 + \omega_3^2$$
$$\omega = \sqrt{\Omega^2 + \omega_3^2}$$

Equations (11) & (12) are parametric equations of circle of radius ω_0. It follows that projection of $\underline{\omega}$ in xy-plane describes a circle with time. Thus $\underline{\omega}$ traces out a cone or precesses a cone about Z-axis with constant angular frequency (angular velocity Ω) while ω_3 remains constant. This motion is relative to body principal axes and to an observer in body coordinate system, $\underline{\omega}$ traces a cone, called body cone whose half angle φ_b is given by



$\omega = \sqrt{\Omega^2 + \omega_3^2}$ where circle is of radius

$\omega_0 = \omega \sin\varphi_b$, $\omega_3 = \omega\cos\varphi_b$

So $\tan\varphi_b = \dfrac{\omega_0}{\omega_3} = \dfrac{\sqrt{\omega_1^2 + \omega_2^2}}{\omega_3}$.

Also the frequency of precession Ω is given by

$\omega_3 = \omega\cos\varphi_b$. $\qquad \Omega = \dfrac{I_3 - I_1}{I_1}\omega_3$

But $\omega = \sqrt{\Omega^2 + \omega_3^2}$

$$\Omega = \left(\frac{I_3 - I_1}{I_1}\right)\omega\cos\varphi_b = \left(\frac{I_3}{I_1} - 1\right)\omega\cos\varphi_b.$$

Problems

M.I about a line with D.Cs l, m, n is given by

$$I = I_{xx} l^2 + I_{yy} m^2 + I_{zz} n^2 + 2 lm I_{xy} + 2 ln I_{xz} + 2 mn I_{yz}$$

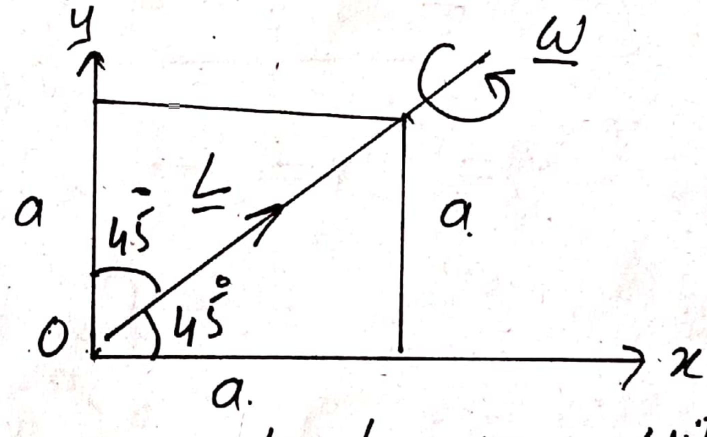
where $I_{xy} = -\sum_i m_i x_i y_i$

$I_{xz} = -\sum_i m_i x_i z_i$

$I_{yz} = -\sum_i m_i y_i z_i$

Question # Find the moment of inertia of a uniform square lamina of side (a) and mass (m) about a diagonal.

Sol



We choose Co-ordinate axes with a corner at the origin and Lamina lying in xy-plane.

$$I_{xx} = \int_{Lamina} y^2 \, dm$$

$$dm = \rho\, dx\, dy$$

$$\rho = \frac{m}{a^2}$$

$$I_{xx} = \int_0^a\int_0^a y^2 \rho\, dy\, dx$$

$$= \rho \int_0^a \left[\frac{y^3}{3}\right]_0^a dx$$

$$= \rho \int_0^a \frac{a^3}{3}\, dx$$

$$= \frac{\rho a^3}{3}\left[x\right]_0^a = \frac{\rho a^4}{3}$$

$$= \frac{m}{a^2}\cdot\frac{a^4}{3} = \boxed{\frac{1}{3} m a^2}$$

By Symmetry

$$\boxed{I_{yy} = \frac{1}{3} m a^2}$$

$$I_{xy} = I_{yx} = -\int_0^a\int_0^a xy\, dm$$

$$= -\rho \int_0^a\int_0^a xy\, dx\, dy$$

$$= -\rho \int_0^a x \left[\frac{y^2}{2}\right]_0^a dx$$

$$= -\rho \int_0^a \frac{a^2}{2} x\, dx$$

$$= -\frac{\rho a^2}{2}\left[\frac{x^2}{2}\right]_0^a = -\frac{\rho a^4}{4}$$

$I_{xy} = I_{yx} = -\frac{m}{a^2} \cdot \frac{a^4}{4} = -\frac{1}{4} ma^2$

Since Lamina is in xy-plane.

Therefore $I_{xz} = I_{yz} = 0$

Diagonal makes angles $45°, 45°$ with x-axis and y-axis. So

$l = \cos 45° = \frac{1}{\sqrt{2}} = m.$

M.I about diagonal is given by

$$I_{diagonal} = I_{xx} l^2 + I_{yy} m^2 + 2lm I_{xy}$$

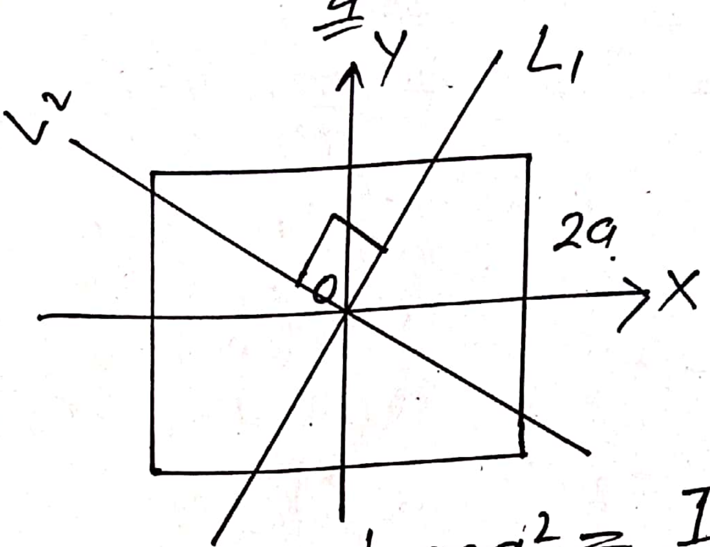
$$= \frac{1}{3} ma^2 \left(\frac{1}{\sqrt{2}}\right)^2 + \frac{1}{3} ma^2 \left(\frac{1}{\sqrt{2}}\right)^2 + 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{4} ma^2\right)$$

$$= \frac{1}{6} ma^2 + \frac{1}{6} ma^2 - \frac{1}{4} ma^2$$

$$\boxed{I = \frac{1}{12} ma^2}$$

Question # Find the M.I of a uniform square plate ~~through~~ about any axis through (any) its centre and lying in the plane of plate

Sol Let O be centre of square plate of side $2a$



Then $I_x = \frac{1}{3} ma^2 = I_y$

and by $\perp$ar axes Theorem

$$I_z = I_x + I_y = \frac{2}{3} ma^2$$

Let L_1 be any line in the plane of plate and passing through its centre. Let L_2 be another line through centre and $\perp$ar to L_1

By $\perp$ar axes Theorem

$$I_{L_1} + I_{L_2} = I_z$$

By symmetry

$$I_{L_1} = I_{L_2}$$

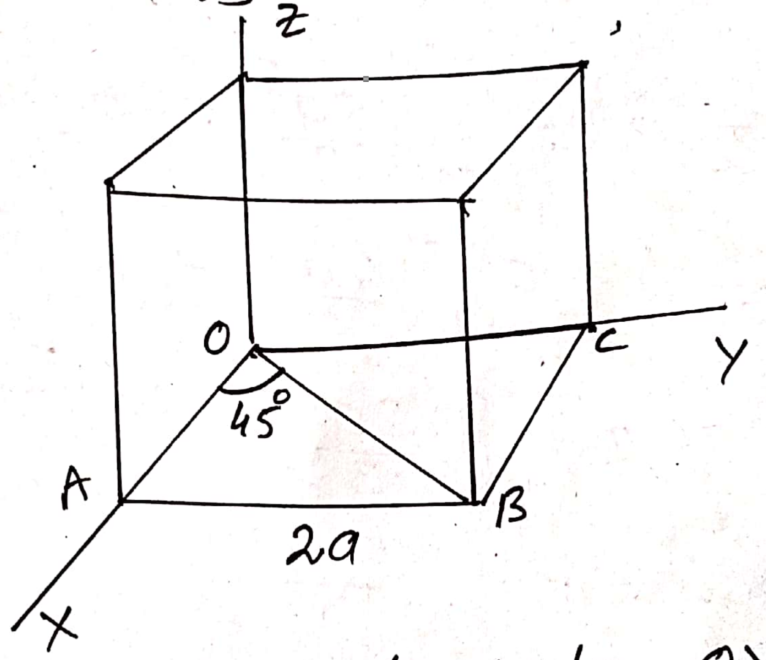
$$\Rightarrow \quad 2 I_{L_1} = I_z = \frac{2}{3} ma^2$$

$$I_{L_1} = \frac{1}{3} ma^2.$$

Here L_1 is arbitrary. Thus M.I about all lines through centre of uniform square lamina is same

Question # 5 Prove that M.I solid cube of side $2a$ about a diagonal of face is $\frac{5}{3} Ma^2$

Sol



Consider co-ordinates system OXYZ with origin O at corner of cube. OB is diagonal face OABC about which M.I is to be calculated.

$$I_{xx} = \int_{cube} (y^2+z^2)\, dm.$$

$$dm = \rho\, dx\, dy\, dz$$

where $\rho = \frac{M}{(2a)(2a)(2a)} = \frac{M}{8a^3}$

$$I_{xx} = \rho \int_0^{2a}\int_0^{2a}\int_0^{2a} (y^2+z^2)\, dx\, dy\, dz$$

$$= \rho \int_0^{2a}\int_0^{2a} (y^2+z^2) \left[x\right]_0^{2a} dy\, dz$$

$$= 2a\rho \int_0^{2a}\int_0^{2a} (y^2+z^2)\,dy\,dz$$

$$= 2a\rho \int_0^{2a} \left[\frac{y^3}{3} + z^2 y\right]_0^{2a} dz$$

$$= 2a\rho \int_0^{2a} \left(\frac{8a^3}{3} + 2a z^2\right) dz$$

$$= 4a^2\rho \left[\frac{4a^2}{3}z + \frac{z^3}{3}\right]_0^{2a}$$

$$= 4a^2\rho \left[\frac{8a^3}{3} + \frac{8a^3}{3}\right]$$

$$= 4a^2\rho \left[\frac{16a^3}{3}\right]$$

$$= 4a^2 \frac{M}{8a^3}\left(\frac{16a^3}{3}\right) = \frac{8}{3}Ma^2$$

$$\boxed{I_{xx} = \tfrac{8}{3}Ma^2}$$

By Symmetry

$$I_{yy} = \tfrac{8}{3}Ma^2$$

$$I_{zz} = \tfrac{8}{3}Ma^2$$

Diagonal makes angles of 45° with x-axis and y-axis

$$l = \cos\alpha = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$m = \cos\beta = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$I_{xz} = -\rho \int_0^{2a}\int_0^{2a}\int_0^{2a} xy \, dx\, dy\, dz$$

$$= -2a\rho \int_0^{2a}\int_0^{2a} xy \, dx\, dy$$

$$= -2a\rho \int_0^{2a} y \left[\frac{x^2}{2}\right]_0^{2a} dy$$

$$= -2a\rho \int_0^{2a} 2a^2 y \, dy$$

$$= -2a\rho \cdot 2a^2 \left[\frac{y^2}{2}\right]_0^{2a}$$

$$= -2a\rho \cdot 2a^2 \cdot 2a^2$$

$$= -8a^5\rho = -8a^5 \frac{M}{8a^3}$$

$$= -Ma^2$$

M.I about diagonal of face is

$$I_{OB} = I_{xx}l^2 + I_{yy}m^2 + 2lm\, I_{xy} + 0 + 0$$

$$= \frac{8}{3}Ma^2\left(\frac{1}{\sqrt{2}}\right)^2 + \frac{8}{3}Ma^2\left(\frac{1}{\sqrt{2}}\right)^2$$

$$+ 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)(-Ma^2)$$

$$= \frac{4}{3}Ma^2 + \frac{4}{3}Ma^2 - Ma^2$$

$$= \frac{8}{3}Ma^2 - Ma^2 = \frac{5}{3}Ma^2$$

Result $= \frac{5}{3}Ma^2$

Directions of Principal Axes in plane

If θ is the angle of a principal axis of plane lamina with x-axis at any point of lamina, then

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2 I_{xy}}{I_{yy} - I_{xx}}\right)$$

where $I_{xy} = -\sum_{i=1}^{n} m_i x_i y_i$

when $I_{xy} = \sum_{i=1}^{n} m_i x_i y_i$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2 I_{xy}}{I_{xx} - I_{yy}}\right)$$

and direction of the other principal axis is $\pi/2 + \theta$ and then M.I about these axes is found by

$$\boxed{I = I_{xx} l^2 + I_{yy} m^2 + 2 l m I_{xy}}$$

Question # For a uniform rectangular lamina (or thin plate) ABCD with sides $2a, 2b$ $(b > a)$, find the directions of principal Axes at a corner A.

Sol

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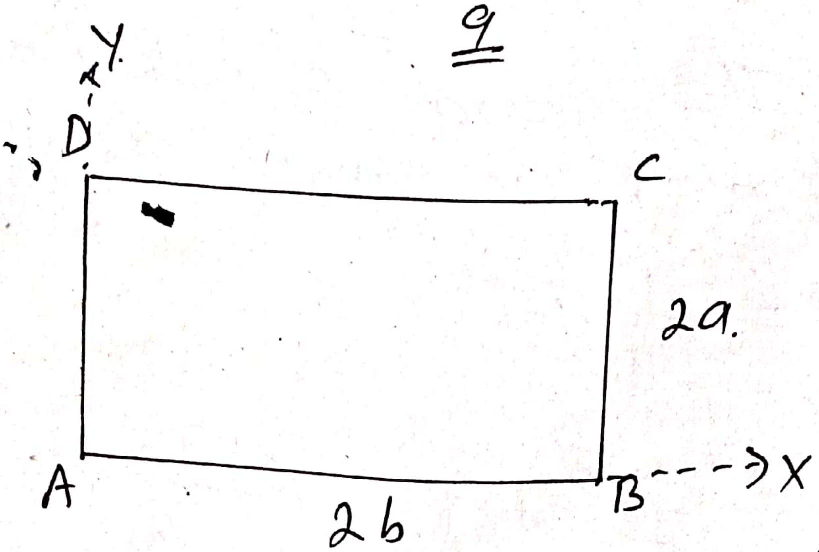


~~Since~~, the directions of pricipal axes is to be found about Corner A, we axes at A.

$$I_{xx} = \rho \int_0^{2b}\int_0^{2a} y^2\,dy\,dx$$

$$= 2b\rho \int_0^{2a} y^2\,dy$$

$$= 2b\rho\left[\frac{y^3}{3}\right]_0^{2a} = 2b\rho\cdot\frac{8a^3}{3}$$

$$= \frac{16}{3}ba^3\rho = \frac{16}{3}ba^3\frac{M}{4ab}$$

$$= \tfrac{4}{3}Ma^2$$

Similarly

$$I_{yy} = \tfrac{4}{3}Mb^2$$

$$I_{xy} = -\rho\int_0^{2b}\int_0^{2a} xy\,dy\,dx$$

$$= -\rho\left[\frac{x^2}{2}\right]_0^{2b}\int_0^{2a} y\,dy$$

$$= -\rho(2b^2)(2a^2) = -4\rho a^2b^2$$

$$= -4\frac{M}{4ab}a^2b^2 = -Mab$$

The direction θ with x-axis at A of one principal axis given by

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2I_{xy}}{I_{yy}-I_{xx}}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2(-Mab)}{4/3\,Mb^2 - 4/3\,Ma^2}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{-2ab}{4/3\,(b^2-a^2)}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{3ab}{2(b^2-a^2)}\right)$$

$$\boxed{\tan 2\theta = \frac{3ab}{2(b^2-a^2)}} \longrightarrow \text{Result}$$

Cor If Lamina is a square of $2a$, then

$$\tan 2\theta = \frac{3aa}{2(0)}$$

$$2\theta = \pi/2$$

$$\theta = \pi/4$$

$\Rightarrow$ one principal axis is along diagonal at A and other is $\perp$ar to it and also along the other diagonal BD

Angular Momentum.

$\underline{L} = \{I\}[\underline{\omega}]$

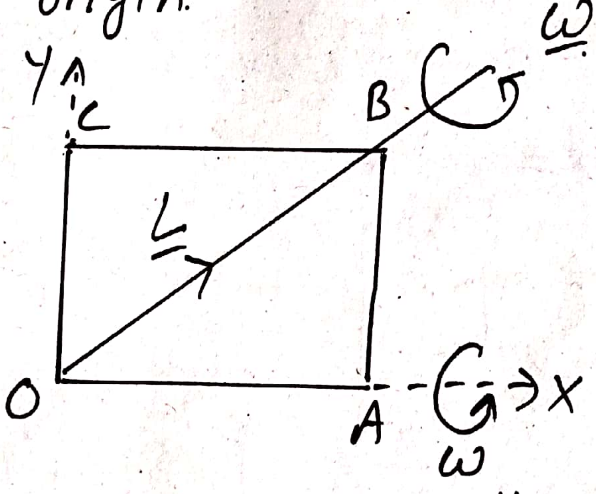
where $\{I\} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$

Question # Find the angular momentum about corner of squar plate. of side (a) when it rotates with angular speed ω about

(a) one of its sides (x-axis)

b) About the diagonal through origin.

Sol



Consider axes OXY with origin at corner as shown in fig

$I_{xx} = \frac{1}{3}ma^2$

$I_{yy} = \frac{1}{3}ma^2$

$I_{zz} = I_{xx} + I_{yy} = \frac{1}{3}ma^2 + \frac{1}{3}ma^2$

$= \frac{2}{3}Ma^2$

$$I_{xy} = I_{yx} = -\frac{1}{4}ma^2 = -\frac{1}{4}ma^2$$

Since lamina is in xy-plane,

$$I_{xz} = 0 = I_{yz}$$

$$\{I\} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$= ma^2 \begin{bmatrix} \frac{1}{3} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 1/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix}$$

(a) For rotation about x-axis (Side OA)

$\omega_x = \omega \qquad \omega_y = 0 \qquad \omega_z = 0$

$$\underline{\omega} = \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{L} = \{I\}\vec{\omega} = ma^2 \begin{bmatrix} 1/3 & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 1/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix}$$

$$= ma^2\omega \begin{bmatrix} 1/3 \\ -1/4 \\ 0 \end{bmatrix}$$

The magnitudes of angular momentum is given by

$$L^2 = (ma^2\omega)^2\left(\frac{1}{9}+\frac{1}{16}\right)$$

$$= (ma^2\omega)^2\left(\frac{16+9}{144}\right)$$

$$= (ma^2\omega)^2\left(\frac{25}{144}\right)$$

$$L = \frac{5}{12}(ma^2\omega)$$

(b) For Rotation About Diagonal OB

The components of $\underline{\omega}$ about the diagonal are.

$$\omega_x = \omega_y = \omega\cos 45^\circ = \frac{\omega}{\sqrt{2}} \quad \omega_z = 0$$

Therefore.

$$\underline{L} = ma^2\begin{bmatrix} \frac{1}{3} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}\begin{pmatrix} \frac{\omega}{\sqrt{2}} \\ \frac{\omega}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$= \frac{ma^2\omega}{\sqrt{2}}\begin{bmatrix} \frac{1}{12} \\ \frac{1}{12} \\ 0 \end{bmatrix}$$

The magnitude of the angular momentum is

$$L^2 = \frac{(ma^2\omega)^2}{2}\left(\frac{1}{144}+\frac{1}{144}\right)$$

$$= \frac{(ma^2\omega)^2}{2}\left(\frac{2}{144}\right) \Rightarrow L = \frac{(ma^2\omega)}{12}.$$

K.E of Rotation

$$T_{rot} = \frac{1}{2}\,\underline{\omega}\cdot\underline{L} \longrightarrow ①$$

$A\cdot B = AB^t$

But

$$\underline{L} = \{I\}[\underline{\omega}]$$

$$\boxed{T_{rot} = \frac{1}{2}[\omega]^t\{I\}[\underline{\omega}]}$$

$$= \frac{1}{2}[\omega_x, \omega_y, \omega_z]\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$= \frac{1}{2}\big(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2I_{xy}\omega_x\omega_y + 2I_{yz}\omega_y\omega_z + 2I_{xz}\omega_x\omega_z\big)$$

Question # Find the rotation K.E of square plate when it is rotating with angular speed ω about

(a) one side. (x-axis)

(b) About diagonal through origin.

Sol #

$$\{I\} = ma^2\begin{bmatrix} \frac{1}{3} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 1/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix}$$

as found above.

(a) For rotation about OA (x-axis)

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$\omega_x = \omega \qquad \omega_y = 0 \qquad \omega_z = 0$

$$[\omega] = \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix}$$

$$T = \frac{1}{2} m a^2 [\omega, 0, 0] \begin{bmatrix} 1/3 & -1/4 & 0 \\ -1/4 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} \omega \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m a^2 \omega^2 [1, 0, 0] \begin{bmatrix} 1/3 & -1/4 & 0 \\ -1/4 & 1/3 & 0 \\ 0 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m a^2 \omega^2 [1, 0, 0] \begin{bmatrix} 1/3 \\ -1/4 \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m a^2 \omega^2 (1/3) = \frac{1}{6} m a^2 \omega^2.$$

(b) For Rotation About Diagonal OB

$$\omega_x = \omega_y = \frac{\omega}{\sqrt{2}}$$

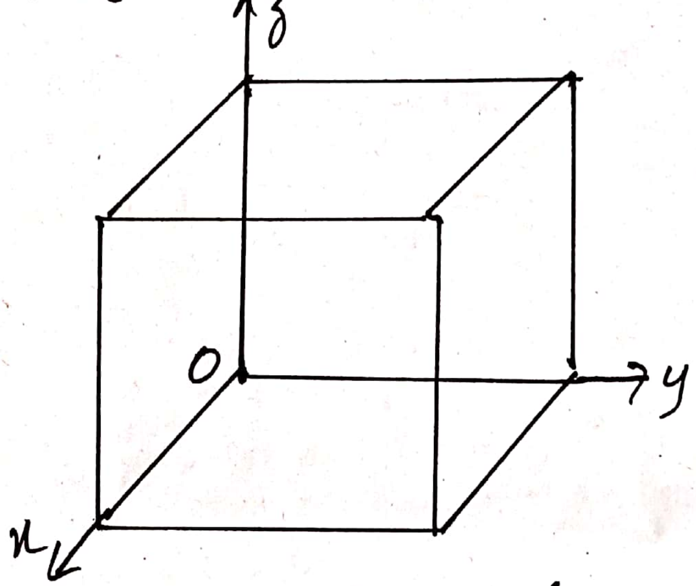
$$T = \frac{1}{2} m a^2 \left[\frac{\omega}{\sqrt{2}}, \frac{\omega}{\sqrt{2}}, 0\right] \begin{bmatrix} 1/3 & -1/4 & 0 \\ -\frac{1}{4} & \frac{1}{3} & 0 \\ 0 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} \frac{\omega}{\sqrt{2}} \\ \frac{\omega}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m a^2 \omega^2 \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right] \begin{bmatrix} \frac{1}{3} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{3} & 0 \\ 0 & 0 & 2/3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$T = \frac{1}{24} m a^2 \omega^2.$$

Question # Find the angular momentum and K.E of rotation of a cube about the point of intersection O of three edges if the cube has angular velocity $\vec{\omega} = 2\hat{i}+5\hat{j}-3\hat{k}$ about O

Sol #



Consider co-ordinates system with origin at the point of intersection of edges O. Let $2a$ be side of cube.

$$I_{xx} = \rho \int_0^{2a}\int_0^{2a}\int_0^{2a} (y^2+z^2)\,dx\,dy\,dz$$

$$= \rho \int_0^{2a}\int_0^{2a} (y^2+z^2)\,[x]_0^{2a}\,dy\,dz$$

$$= 2a\rho \int_0^{2a}\int_0^{2a} (y^2+z^2)\,dy\,dz$$

$$= 2a\rho \int_0^{2a} \left[\frac{y^3}{3}+z^2y\right]_0^{2a} dz$$

$= 4a^2\rho\left[\frac{4a^2}{3}z + \frac{z^3}{3}\right]_0^{2a}$

$= 8/3\, ma^2$

By symmetry

$I_{yy} = 8/3\, ma^2$

$I_{zz} = 8/3\, ma^2$

$I_{xz} = -\rho \int_0^{2a}\int_0^{2a}\int_0^{2a} xz\, dx\, dy\, dz$

$= -2a\rho \int_0^{2a}\int_0^{2a} xz\, dx\, dz$

$= -2a\rho \int_0^{2a} \left[\frac{x^2}{2}\right]_0^{2a} z\, dz$

$= -2a\rho \cdot 2a^2 \int_0^{2a} z\, dz$

$= -2a\rho \cdot 2a^2 \cdot 2a^2$

$= -8a^5\rho = -8a^5 \frac{m}{8a^3}$

$= -ma^2$

By symmetry

$I_{xy} = -ma^2 = I_{yz}$

Hence M.I matrix is given by

$$[I] = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}$$

$$[I] = ma^2 \begin{bmatrix} 8/3 & -1 & -1 \\ -1 & 8/3 & -1 \\ -1 & -1 & 8/3 \end{bmatrix}$$

$$\vec{\omega} = 2\hat{i} + 5\hat{j} - 3\hat{k}$$

$\omega_x = 2 \quad \omega_y = 5 \quad \omega_z = -3$

$$[L] = [I][\omega]$$

$$= ma^2 \begin{bmatrix} \frac{8}{3} & -1 & -1 \\ -1 & 8/3 & -1 \\ -1 & -1 & 8/3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$$

$$= ma^2 \begin{bmatrix} \frac{16}{3} - 5 + 3 \\ -2 + \frac{40}{3} + 3 \\ -2 - 5 - \frac{24}{3} \end{bmatrix}$$

$$= ma^2 \begin{bmatrix} 10/3 \\ \frac{43}{3} \\ -45/3 \end{bmatrix}$$

Magnitude of angular momentum is

$$L^2 = (ma^2)^2 \left(\frac{100}{9} + \frac{(43)^2}{9} + \frac{(45)^2}{9} \right)$$

$$= (ma^2)^2 \left(\frac{39.74}{9} \right)$$

$$L = (21.01) ma^2$$

Rotational K.E is given by

$$T_{rot} = ma^2 [2, 5, -3] [I] \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$$

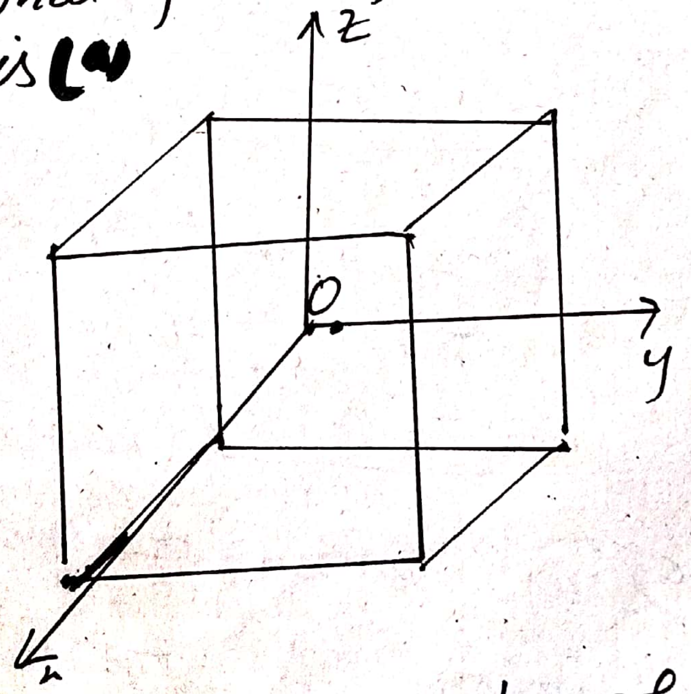
$$= ma^2 [2, 5, -3] \begin{bmatrix} 10/3 \\ 43/3 \\ -45/3 \end{bmatrix}$$

$$= ma^2 \left[\frac{20}{3} + \frac{215}{3} + \frac{135}{3} \right]$$

$$T_{rot} = ma^2 \left(\frac{370}{3} \right)$$

Question # Find the moment of inertia about a diagonal of a uniform cube whose side length is (a)

Sol.



Let O be centre of cube of side a.

$$I_{xx} = \rho \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} (y^2 + z^2)\, dx\, dy\, dz$$

$I_{xx} = \frac{1}{6} ma^2.$

By Symmetry

$I_{yy} = \frac{1}{6} ma^2$

$I_{zz} = \frac{1}{6} ma^2$

$I_{xy} = 0 = I_{xz} = I_{yz}$

Now any diagonal of cube passes through O. Let l, m, n be its D.Cs.

M.I about diagonal is

$I_{diagonal} = I_{xx} l^2 + I_{yy} m^2 + I_{zz} n^2 + 0 + 0 + 0$

$= \frac{1}{6} ma^2 l^2 + \frac{1}{6} ma^2 m^2 + \frac{1}{6} ma^2 n^2$

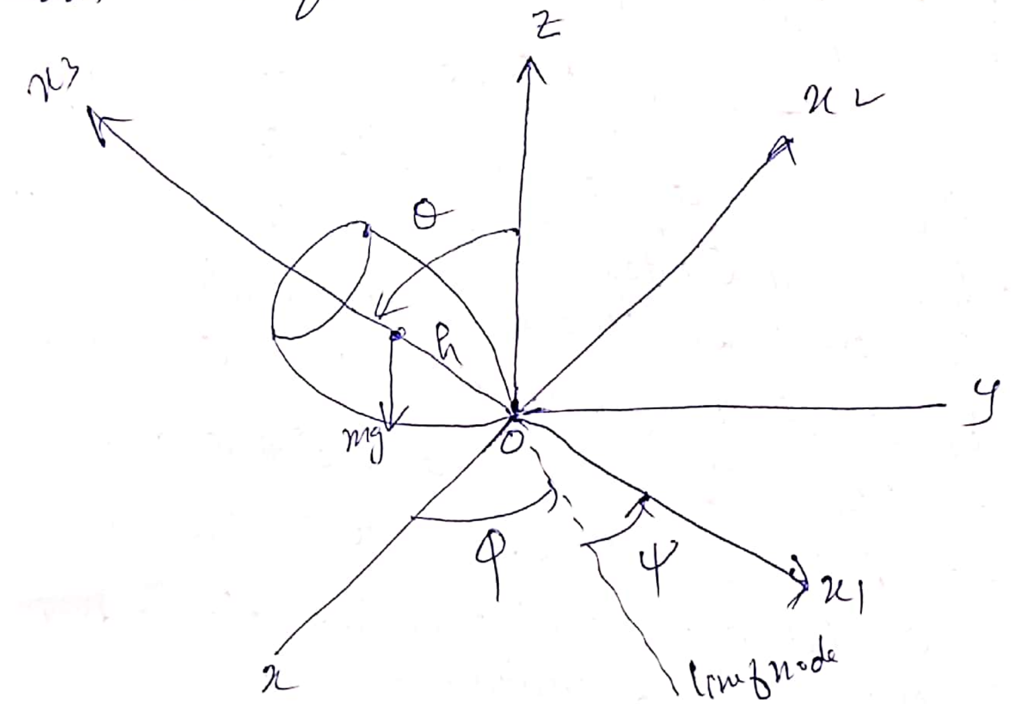
$= \frac{1}{6} ma^2 (l^2 + m^2 + n^2)$

$= \boxed{\frac{1}{6} ma^2}.$

Thus moment of inertia about any diagonal is $\frac{1}{6} ma^2.$

Motion of Symmetric Top with one Point Fixed

Consider a symmetric top with tip held fixed. rotating in a gravitational field. we can seperate translational & rotational K.E by taking c.m as origin of body axes. Also if the origins of the fixed and body co-ordinate systems coincide, then traslational K.E vanish. Such a choice is quite convenient for discussing the top. because tip of the top can be taken as origin of both systems. let tip of top be origin both fixed system $Oxyz$ and body co-ordinate system $Ox_1x_2x_3$. The z-axis (fixed) is vertical and its symmetry axis (x_3-axis) which is its spin axis as well as its principal axis of symmetry is at an angle θ from fixed z-axis. let h be distance of c.m from tip and m be mass of top. let I_3 be the m.o.i about by x_3-axis and the principal moments of inertia about x_1-axis & x_2-axis are equal $I_1 = I_2 = I$, where $I_3 \neq I$



K.E

K.E of the top is given by

$$T = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2$$

$$= \frac{1}{2} I_1 (\omega_1^2 + \omega_2^2) + \frac{1}{2} I_3 \omega_3^2 \longrightarrow (1)$$

But In terms of Eulers angles

$$\omega_1 = \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi$$

$$\omega_2 = \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi$$

$$\omega_3 = \dot{\phi} \cos\theta + \dot{\psi}$$

using these in (1)

$$T = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi} \cos\theta + \dot{\psi})^2 \longrightarrow (2)$$

$$P.E = mgh \cos\theta = V$$

The Lagrangian of Motion is

$$L = T - V$$

$$L = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3 (\dot{\phi} \cos\theta + \dot{\psi})^2 - mgh\cos\theta \longrightarrow (3)$$

note that in Lagrangian. the Co-ordinates used other than Cartesian Co-ordinates which may be angles or other quantites are called generalised Coordinates and derivatives of L w.r.t. Co-ordinates are called generalised momenta relative to these Co-ordinates. Now if a co-ordinate is absent in L, then it is called cyclic and momentum relative to it is necessarily Const of motion.

We note L is cyclic in both the φ- and ψ- Co-ordinates which are angles, their relative momenta will be angular momenta along the axis for φ, ψ are rotation angles i.e z-(vertical) axis and x_3 - (or body symmetry axis).

The momenta conjugate to both φ, ψ are Constant of motion.

$$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = (I_1 \sin^2\theta + I_3 \cos^2\theta)\dot\varphi + I_3 \dot\psi \cos\theta$$

$$= \text{Constant} = C_1 \longrightarrow (4)$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\varphi \cos\theta) = \text{Constant} = C_2 \rightarrow (5)$$

From (5)

$$\dot\psi = \frac{C_2 = p_\psi - I_3 \dot\varphi \cos\theta}{I_3} \longrightarrow (6)$$

using this in (4) we find

$$(I_1 \sin^2\theta + I_3 \cos^2\theta)\dot\varphi + I_3\left(\frac{C_2 - I_3 \dot\varphi \cos\theta}{I_3}\right)\cos\theta = C_1 = p_\varphi$$

$$(I_1 \sin^2\theta)\dot\varphi + C_2 \cos\theta = C_1 = p_\varphi$$

$$\dot\varphi = \frac{C_1 = p_\varphi - C_2 \cos\theta}{I_1 \sin^2\theta} \longrightarrow (7)$$

using (7) in (6)

$$\dot\psi = \frac{C_2}{I_3} - \frac{(C_1 - C_2\cos\theta)\cos\theta}{I_1 \sin^2\theta} \longrightarrow (8)$$

Since the system we are considering is Conservative, therefore total Energy of the system is Constant.

Total Energy = E = T + V

$$E = \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + \frac{1}{2} I_3 \omega_3^2 + mgh\cos\theta \longrightarrow (9)$$

= Constant = C_3

From (5) using expression for ω_3

$$P_\psi = I_3 \omega_3 = C_2 \longrightarrow (10)$$

or $I_3^2 \omega_3^2 = C_2^2$

$$I_3 \omega_3^2 = \frac{C_2^2}{I_3} = \frac{P_\psi^2}{I_3} = \text{Constant} = C_4 \longrightarrow (11)$$

Thus not only E is a Constant of Motion but $E - \frac{1}{2} I_3 \omega_3^2$ is Constant of Motion.

Let $E' = E - \frac{1}{2} I_3 \omega_3^2 \longrightarrow (12)$

$$= \frac{1}{2} I_1 (\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2) + mgh\cos\theta \longrightarrow (12)$$

= Constant = C_5

putting value of $\dot{\phi}$ from (7), we have

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{(P_\phi - P_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + mgh\cos\theta \longrightarrow (13)$$

which can be written as

$$E' = \frac{1}{2} I_1 \dot{\theta}^2 + V(\theta) \longrightarrow (14)$$

when $V(\theta)$ is effective potential given by

$$V(\theta) = \frac{(P_\phi - P_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + mgh\cos\theta \longrightarrow (15)$$

From (15)

$$\dot{\theta}^2 = \frac{2}{I_1}[E' - V(\theta)]$$

$$\dot{\theta} = \sqrt{\frac{2}{I_1}[E' - V(\theta)]}$$

$$\frac{d\theta}{dt} = \frac{2}{I_1}[E' - V(\theta)]$$

$$\Rightarrow dt = \frac{1}{\frac{2}{I_1}(E' - V(\theta))} d\theta$$

$$t(\theta) = \int \frac{d\theta}{\sqrt{\frac{2}{I_1}[E' - V(\theta)]}} \rightarrow (16)$$

from (16) we can find $\theta(t)$ and then $\varphi(t)$, $\psi(t)$ in terms of E', P_φ, t. This can be obtained in terms of E', P_φ, t. This can give the motion of symmetric top (configuration) at any time t.

We discuss motion the motion using effective potential

Let $u = \cos\theta$

$$V' = \frac{(P_\varphi - P_\psi u)^2}{2I_1(I - u^2)} + mghu$$

V' becomes infinite at $u = 1$ or $\theta = 0$. and at $u = -1$ or $\theta = \pi$. Hence V' has minimum value in between $u = -1$ and $u = 1$. For getting min. point

$$\frac{\partial V'}{\partial u} = \frac{(P_\varphi - P_\psi u)(P_\psi - P_\varphi u)}{I_1(1 - u^2)^2} + mgl$$

Let V' be minimum at $u = u_0$ $(\theta = \theta_0)$
Then at $u = u_0$ we have.

$$\left.\frac{\partial V'}{\partial u}\right|_{u=u_0} = 0$$

$$\Rightarrow mgh(1-u_0^2)^2 = \frac{(P_\varphi - P_\psi u_0)(P_\psi - P_\varphi u_0)}{I_1} \rightarrow (17)$$

Then from equation (7), the $\dot\varphi$ at min points

$$\dot\varphi_0 = \frac{P_\varphi - P_\psi \cos\theta_0}{I_1 \sin^2\theta_0} = \frac{P_\varphi - P_\psi u_0}{I_1(1-u_0^2)} \rightarrow (18)$$

$\dot\varphi_0$ is is the precessional velocity of the top when inclination with space axis (3-axis) is θ_0

Equation (18 can be written as

$$u_0 P_\varphi^2 - (1+u_0^2) P_\psi P_\varphi + u_0 P_\psi^2 + mgh I_1 (1-u_0^2)^2 = 0$$

It is quadratic in P_φ function.

$$P_\varphi = \frac{(1+u_0^2)P_\psi \pm (1-u_0^2)\sqrt{P_\psi^2 - 4u_0 MghI_1}}{2u_0}$$

$$= \frac{(1-u_0^2)P_\psi}{2u_0}\left[1 \pm \sqrt{1 - \frac{4MghI_1 u_0}{P_\psi^2}}\right] + u_0 P_\psi$$

we get

$$(P_\varphi - u_0 P_\psi) = \frac{(1-u_0^2)P_\psi}{2u_0}\left[1 \pm \sqrt{1 - \frac{4MghI_1 u_0}{P_\psi^2}}\right]$$

$P_\varphi - u_0 P_\psi$ is real for for fixed θ_0 such that $\theta_0 < \pi/2$ or $u_0 < 1$

Now $p_\psi = I_3 \omega_3$

$$\omega_3 = \frac{p_\psi}{I_3} \longrightarrow (19)$$

When $\theta_0 < \pi/2$, the min value of p_ψ is

$$1 - \frac{4Mgh I_1 \cos\theta_0}{p_{\psi\,min}^2} = 0$$

$$p_{\psi\,min}^2 = 4mgh I_1 \cos\theta_0$$

from (19)

$$\omega_{3\,min}^2 = \frac{4Mgh I_1 \cos\theta_0}{I_3^2}$$

When $E' = V'(\theta_0)$ from (14)

$\dot{\theta} = 0 \Rightarrow \theta = \text{Constant}$

and top does not nutate.

For $\theta = 0$ equation (14) has two roots θ_1 & θ_2. These two roots are turning points and they decide the behavior of tip of the z-axis of top. When θ varies b/w θ_1 & θ_2 $\dot{\phi}$ will change sign when $(p_\phi - p_\psi \cos\theta)$ changes sign with range of θ